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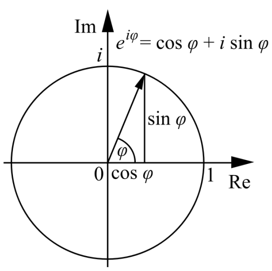
June 28, 2011

Discrete Fourier Transform Handout

1. Euler's Formula

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \tag{1}$$



$$\begin{aligned}
z &= x + iy = |z|(\cos\varphi + i\sin\varphi) = re^{i\varphi} && \text{(vector)}\\
\bar{z} &= x - iy = |z|(\cos\varphi - i\sin\varphi) = re^{-i\varphi} && \text{(complex conjugate)}\\
x &= \mathrm{Re}(z) && \text{(real part)}\\
y &= \mathrm{Im}(z) && \text{(imaginary part)}\\
r &= |z| = \sqrt{x^2 + y^2} && \text{(amplitude)}\\
\varphi &= \arg(z) = \operatorname{atan2}(y, x) && \text{(angle)}
\end{aligned}$$

2. Discrete Fourier Transform

$$\mathcal{F}(\mathrm{x}) = X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}kn} \qquad k = 0, \ldots, N-1 \tag{2}$$

x is an array of data
N is the size of the array (sample size)
X is an array of complex numbers the same size as x

3. Arguments of Polar Form of Fourier Coefficients

The polar form of a complex number is in the form (A, φ), where A is the amplitude and φ is the phase (angle). The following are the formulas for finding the arguments of the polar form of the Fourier coefficients.

$$A_k = |X_k| = \sqrt{\mathrm{Re}(X_k)^2 + \mathrm{Im}(X_k)^2} \tag{3}$$

$$\varphi_k = \arg(X_k) = \operatorname{atan2}(\mathrm{Im}(X_k), \mathrm{Re}(X_k)) \tag{4}$$

4. Example DFT of Set [0146]

$x = \{1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0\}$
N = size of x = 12

$$\begin{aligned} X_0 &= \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}0n} \\ &= e^{-\frac{2\pi i}{N}0*0} + e^{-\frac{2\pi i}{N}0*1} + e^{-\frac{2\pi i}{N}0*4} + e^{-\frac{2\pi i}{N}0*6} \\ &= 1+1+1+1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} X_1 &= \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}1n} \\ &= e^{-\frac{2\pi i}{N}1*0} + e^{-\frac{2\pi i}{N}1*1} + e^{-\frac{2\pi i}{N}1*4} + e^{-\frac{2\pi i}{N}1*6} \\ &= 1 + (\tfrac{\sqrt{3}}{2} - \tfrac{i}{2}) + (-\tfrac{1}{2} - \tfrac{i\sqrt{3}}{2}) - 1 \\ &= \tfrac{\sqrt{3}-1}{2} - \tfrac{i\sqrt{3}+1}{2} \end{aligned}$$

$$\vdots$$

$$\begin{aligned} X_{11} &= \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N}11n} \\ &= e^{-\frac{2\pi i}{N}11*0} + e^{-\frac{2\pi i}{N}11*1} + e^{-\frac{2\pi i}{N}11*4} + e^{-\frac{2\pi i}{N}11*6} \\ &= 1 + (\tfrac{\sqrt{3}}{2} + \tfrac{i}{2}) + (-\tfrac{1}{2} + \tfrac{i\sqrt{3}}{2}) - 1 \\ &= \tfrac{\sqrt{3}-1}{2} + \tfrac{i\sqrt{3}+1}{2} \end{aligned}$$

$$\begin{aligned} X_k &= \{4,\ \frac{\sqrt{3}-1}{2} - \frac{i\sqrt{3}+1}{2},\ 2,\ 1-i,\ 1-i\sqrt{3},\ -\frac{\sqrt{3}+1}{2} + \frac{i\sqrt{3}-1}{2},\ 2, \\ &\quad -\frac{\sqrt{3}+1}{2} - \frac{i\sqrt{3}-1}{2},\ 1+i\sqrt{3},\ 1+i,\ 2,\ \frac{\sqrt{3}-1}{2} + \frac{i\sqrt{3}+1}{2}\} \\ A_k &= \{4,\ \sqrt{2},\ 2,\ \sqrt{2},\ 2,\ \sqrt{2},\ 2,\ \sqrt{2},\ 2,\ \sqrt{2},\ 2,\ \sqrt{2}\} \\ \varphi_k &= \{0,\ -\frac{5\pi}{12},\ 0,\ -\frac{3\pi}{12},\ -\frac{4\pi}{12},\ \frac{11\pi}{12},\ 0,\ -\frac{11\pi}{12},\ \frac{4\pi}{12},\ \frac{3\pi}{12},\ 0,\ \frac{5\pi}{12}\} \end{aligned}$$