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Pumping the All-Interval Tetrachords: Algorithms for Generating Z-Related Sets

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Figure 1: General formula for z-related sets, covering 12 of the 23 z-related pairs in mod12 (adapted from Callender and Hall 2008).

$$\Phi \cup x + \Psi \quad \zeta \quad \Phi \cup -x + \Psi \qquad \bmod m$$

ζ	–	Indicates z-relation
Φ	–	Cyclic pc-set in $\mathbb{R}_m$ with cycle interval ϕ, containing 0, such as $\{0,6\}$ or $\{0,4,8\}$ in mod12
Ψ	–	Non-cyclic pc-set with certain conditions (see Appendix)
x	–	Real number $\not\equiv 0 \bmod \phi/2$

Figure 2: Example formula in mod12, where $\phi = 6$ and the cyclic and non-cyclic sets are {0,6} and {0,3}, which forms the all-interval tetrachords when x is an integer and $x \not\equiv 0$ mod3.

$\Phi = \{0,6\}$
$\Psi = \{0,3\}$

$$\{0,6\} \cup x + \{0,3\} \quad \zeta \quad \{0,6\} \cup -x + \{0,3\} \qquad \text{mod } 12$$

EXAMPLE: $x = 1$

$$\begin{aligned} \{0,6\} \cup 1 + \{0,3\} \quad &\zeta \quad \{0,6\} \cup -1 + \{0,3\} \\ \{0,6,1,4\} \quad &\zeta \quad \{11,0,2,6\} \\ \text{sc}\,[0146] \quad &\zeta \quad \text{sc}\,[0137] \end{aligned}$$

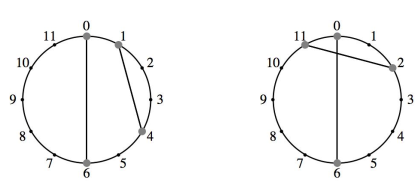

Figure 3: The 12 z-related setclass pairs in $\mathbb{Z}_{12}$ that can be generated by Callender and Hall's formula.

(a) Tetrachords

[0146]	[0137]	[06] + [03]

(b) Pentachords

[01356]	[01247]	[06] + [024]
[01457]	[01258]	[06] + [015]
[01348]	[03458]	[048] + [02]

(c) Hexachords

[012356]	[012347]	[06] + [0124]
[012456]	[012348]	[06] + [0134]
[013457]	[023458]	[06] + [0125]
[012457]	[012358]	[06] + [0135]
[013468]	[012469]	[06] + [0237]
[013568]	[012479]	[06] + [0247]
[013478]	[012569]	[06] + [0148]
[013578]	[012579]	[06] + [0358]

Table 1: Five pumping algorithms for the all-interval tetrachords.

$\Phi = \{0, 6\}$, $\Psi = \{0, 3\}$
Let $\alpha = \Phi \cup 1 + \Psi = \{0, 1, 4, 6\}$
Let $\beta = \Phi \cup -1 + \Psi = \{11, 0, 2, 6\}$

p1: Let $Y \subseteq \mathbb{R}_{12}$ be a chosen set of real numbers. For each $y_n \in Y$, add $y_n + \Phi$ to α, and $-y_n + \Phi$ to β.

$$\alpha \cup y_0 + \Phi \cup \ldots \cup y_n + \Phi \quad \zeta \quad \beta \cup -y_0 + \Phi \cup \ldots \cup -y_n + \Phi$$
$$\alpha \bigcup_{k=0}^{n} y_k + \Phi \quad \zeta \quad \beta \bigcup_{k=0}^{n} -y_k + \Phi$$

EXAMPLE: $Y = \{-1, 2\}$

$$\alpha \cup -1 + \{0, 6\} \cup 2 + \{0, 6\} \quad \zeta \quad \beta \cup 1 + \{0, 6\} \cup -2 + \{0, 6\}$$
$$\{11, 0, 1, 2, 4, 5, 6, 8\} \quad \zeta \quad \{10, 11, 0, 1, 2, 4, 6, 7\}$$
$$\text{sc}\,[01235679] \quad \zeta \quad \text{sc}\,[01234689]$$

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p2: Let $Y \subseteq \mathbb{R}_{12}$ be a chosen set of real numbers. For each $y_n \in Y$, add $y_n + \Psi$ to α, and $-y_n + \Psi$ to β.

$$\alpha \cup y_0 + \Psi \cup \ldots \cup y_n + \Psi \quad \zeta \quad \beta \cup -y_0 + \Psi \cup \ldots \cup -y_n + \Psi$$
$$\alpha \bigcup_{k=0}^{n} y_k + \Psi \quad \zeta \quad \beta \bigcup_{k=0}^{n} -y_k + \Psi$$

EXAMPLE: $Y = \{-4\}$

$$\alpha \cup -4 + \{0, 3\} \quad \zeta \quad \beta \cup 4 + \{0, 3\}$$
$$\{11, 0, 1, 4, 6, 8\} \quad \zeta \quad \{11, 0, 2, 4, 6, 7\}$$
$$\text{sc}\,[012579] \quad \zeta \quad \text{sc}\,[013578]$$

Table 1, cont.

p3: Let $S \subseteq \mathbb{R}_{12}$ be a pc-set that is symmetrical at the index 0 or 6. Add $1+S$ to α, and $-1+S$ to β. (Algorithm derives from O'Rourke, et al. 2008.)

$$\alpha \cup 1+S \quad \zeta \quad \beta \cup -1+S$$

EXAMPLE: $i=0$, $S=\{10,2\}$

$$\begin{array}{rcl} \alpha \cup 1+\{10,2\} & \zeta & \beta \cup -1+\{10,2\} \\ \{11,0,1,3,4,6\} & \zeta & \{9,11,0,1,2,6\} \\ \mathrm{sc}\,[012457] & \zeta & \mathrm{sc}\,[013578] \end{array}$$

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p4: Let $S \subseteq \mathbb{R}_{12}$ be a pc-set that is symmetrical at the index 3 or 9. Add S to α, and $3-S$ to β.

$$\alpha \cup S \quad \zeta \quad \beta \cup 3-S$$

EXAMPLE: $S=\{7,8\}$

$$\begin{array}{rcl} \alpha \cup \{7,8\} & \zeta & \beta \cup 3-\{7,8\} \\ \{0,1,4,6,7,8\} & \zeta & \{6,7,8,11,0,2\} \\ \mathrm{sc}\,[012478] & \zeta & \mathrm{sc}\,[012568] \end{array}$$

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p5: This algorithm begins with $x+\Psi$ on both sides.

$$\Phi \cup x+\Psi \quad = \quad \Phi \cup x+\Psi$$

Given that $\Phi=\{0,6\}$ and $\Psi=\{0,3\}$ in mod12, let $S \subseteq \mathbb{R}_{12}$ be a set that is symmetrical at the index $i=2x-3$. Add S to one side of the formula, and $i+6-S$ to the other.

$$\Phi \cup x+\Psi \cup S \quad \zeta \quad \Phi \cup x+\Psi \cup i+6-S$$

EXAMPLE: $x=1$, $i=11$, $S=\{2,9\}$

$$\begin{array}{rcl} \{0,6\} \cup 1+\{0,3\} \cup \{2,9\} & \zeta & \{0,6\} \cup 1+\{0,3\} \cup 5-\{2,9\} \\ \{0,1,2,4,6,9\} & \zeta & \{0,1,3,4,6,8\} \\ \mathrm{sc}\,[012469] & \zeta & \mathrm{sc}\,[013468] \end{array}$$

Table 2: The 22 z-related pairs as pumped all-interval tetrachords.

$\alpha = \{0, 1, 4, 6\}$
$\beta = \{11, 0, 2, 6\}$

Pentachords

Partition	Setclass	AIT Union	Setclass	AIT Union	Pumping Alg.
[06]+[024]	[01356]	-	[01247]	-	n/a
[06]+[015]	[01457]	-	[01258]	-	n/a
[048]+[02]	[01348]	-	[03458]	-	n/a

Hexachords

Partition	Setclass	AIT Union	Setclass	AIT Union	Pumping Alg.	Notes
[06]+[06]+[03]	[012567]	$\alpha \cup \{11{,}5\}$	[012378]	$\beta \cup \{1{,}7\}$	$p1,\ Y = \{-1\}$	
[06]+[0134]	[012456]	$\alpha \cup \{2{,}5\}$	[012348]	$\beta \cup \{10{,}1\}$	$p2,\ Y = \{2\}$	
[06]+[0358]	[012579]	$\alpha \cup \{8{,}11\}$	[013578]	$\beta \cup \{4{,}7\}$	$p2,\ Y = \{-4\}$	
[06]+[0135]	[012457]	$\alpha \cup \{11{,}3\}$	[012358]	$\beta \cup \{9{,}1\}$	$p3,\ i = 0,\ S = \{10, 2\}$	
[06]+[0148]	[012569]	$\alpha \cup \{5{,}9\}$	[013478]	$\beta \cup \{3{,}7\}$	$p3,\ i = 0,\ S = \{4, 8\}$	
[06]+[0124]	[012356]	$\alpha \cup \{3{,}5\}$	[012347]	$\beta \cup \{1{,}3\}$	$p3,\ i = 6,\ S = \{2, 4\}$	
[06]+[0247]	[012479]	$\alpha \cup \{9{,}11\}$	[013568]	$\beta \cup \{7{,}9\}$	$p3,\ i = 6,\ S = \{8, 10\}$	
[06]+[06]+[04]	[012568]	$\alpha \cup \{5{,}10\}$	[012478]	$\beta \cup \{5{,}10\}$	$p4,\ S = \{5, 10\}$	
		$\beta \cup \{7{,}8\}$		$\alpha \cup \{7{,}8\}$	$p4,\ S = \{7, 8\}$	
[06]+[06]+[02]	[012467]	$\alpha \cup \{2{,}7\}$	[012368]	$\beta \cup \{1{,}8\}$	$p4,\ S = \{2, 7\}$	
		$\beta \cup \{4{,}5\}$		$\alpha \cup \{10{,}11\}$	$p4,\ S = \{4, 5\}$	
[06]+[0125]	[013457]	$\beta \cup \{3{,}4\}$	[023458]	$\beta \cup \{9{,}10\}$	$p5,\ S = \{3, 4\}$	
[06]+[0237]	[013468]	$\alpha \cup \{3{,}8\}$	[012469]	$\alpha \cup \{2{,}9\}$	$p5,\ S = \{3, 8\}$	
[06]+[06]+[01]	[013467]	$\alpha \cup \{3{,}7\}$	[012369]	-	n/a	$\alpha \cup (3 + \{0, 4\})$
		$\beta \cup \{3{,}5\}$				$\beta \cup (3 + \{0, 2\})$
[06]+[06]+[05]	[014679]	$\alpha \cup \{7{,}9\}$	[023679]	-	n/a	$\alpha \cup (-3 - \{0, 2\})$
		$\beta \cup \{5{,}9\}$				$\beta \cup (-3 - \{0, 4\})$
[06]+[06]+[02]	[023568]	$\alpha \cup \{3{,}10\}$	[023469]	-	n/a	$\alpha \cup (3 - \{0, 5\})$
		$\beta \cup \{8{,}9\}$				$\beta \cup (-3 - \{0, 1\})$
[06]+[06]+[04]	[013479]	$\alpha \cup \{9{,}10\}$	[013569]	-	n/a	$\alpha \cup (-3 + \{0, 1\})$
		$\beta \cup \{3{,}8\}$				$\beta \cup (3 + \{0, 5\})$

Heptachords

Partition	Setclass	AIT Union	Setclass	AIT Union	Pumping Alg.
[06]+[06]+[024]	[0123568]	$\alpha \cup \{3, 5, 10\}$	[0123479]	$\beta \cup \{1, 3, 8\}$	$p3,\ i = 6,\ S = \{2, 4, 9\}$
		$\beta \cup \{9, 10, 11\}$		$\alpha \cup \{7, 8, 9\}$	$p3,\ i = 6,\ S = \{8, 9, 10\}$
[06]+[06]+[015]	[0124578]	$\alpha \cup \{3, 7, 11\}$	[0145679]	$\beta \cup \{1, 5, 9\}$	$p3,\ i = 0,\ S = \{0, 4, 8\}$
		$\beta \cup \{3, 5, 7\}$		$\alpha \cup \{5, 7, 9\}$	$p3,\ i = 0,\ S = \{4, 6, 8\}$
[048]+[02]+[06]	[0134578]	$\beta \cup \{4, 5, 8\}$	[0124569]	$\alpha \cup \{2, 5, 9\}$	no algorithm

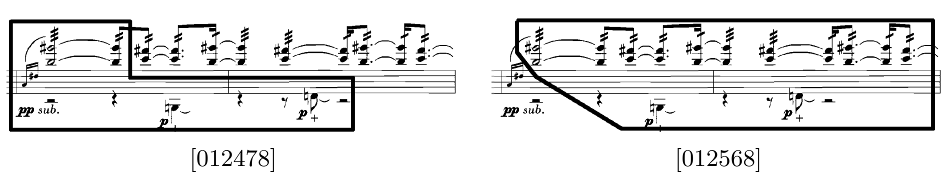
Table 2, cont.

Octachords

Partition	Setclass	AIT Union	Setclass	AIT Union	Pumping Alg.
[06]+[06]+[06]+[03]	[01235679]	$\alpha \cup \{2,5,8,11\}$	[01234689]	$\beta \cup \{1,4,7,10\}$	$p1$, $Y = \{-1, 2\}$, or $p2$, $Y = \{-1, 5\}$
		$\alpha \cup \{3,5,9,11\}$		$\beta \cup \{1,3,7,9\}$	$p1$, $Y = \{-1, 3\}$, or $p3$, $S = \{2, 4, 8, 10\}$
		$\alpha \cup \{2,5,7,10\}$		$\beta \cup \{1,5,8,10\}$	$p2$, $Y = \{-5, 2\}$
		$\alpha \cup \{7,8,10,11\}$		$\beta \cup \{4,5,7,8\}$	$p2$, $Y = \{-5, -4\}$
		$\beta \cup \{3,4,9,10\}$		$\alpha \cup \{2,3,8,9\}$	$p1$, $Y = \{2, 3\}$
		$\beta \cup \{1,4,5,8\}$		$\alpha \cup \{7,10,11,2\}$	$p2$, $Y = \{-1, -5\}$
		$\beta \cup \{5,7,8,10\}$		$\alpha \cup \{5,7,8,10\}$	$p4$, $S = \{5, 7, 8, 10\}$

Figure 4: Carter, Fantasy—Remembering Roger for violin (1999), mm. 55-63.

Figure 5: AITs pumped by open string pizzicati in mm. 56-57, using $p4$ where $S = \{5, 10\}$.



$$
\begin{array}{rcl}
\{7,8,9,11,2,3\} & \zeta & \{6,7,8,11,0,2\} \\
\{8,9,11,3\} \cup \{2,7\} & \zeta & \{6,8,11,0\} \cup \{2,7\} \\
9 + (\beta \cup \{5,10\}) & \zeta & 0 - (\alpha \cup \{5,10\})
\end{array}
$$

Figure 6: Two interpretations of z-related hexachords in opening of Schoenberg's Op. 11, no. 1.

a) Interpreted as [06]+[0125].

$$
\begin{aligned}
\{4,5,7,8,9,11\} \quad &\zeta \quad \{6,8,9,10,11,2\} \\
\{5,11\} \cup \{4,7,8,9\} \quad &\zeta \quad \{2,8\} \cup \{6,9,10,11\} \\
11-(\{0,6\} \cup \{2,3,4,7\}) \quad &\zeta \quad 8-(\{0,6\} \cup \{9,10,11,2\}) \\
11-(\{0,6\} \cup 2.5+\{11.5,0.5,1.5,4.5\}) \quad &\zeta \quad 8-(\{0,6\} \cup -2.5+\{11.5,0.5,1.5,4.5\})
\end{aligned}
$$

b) Interpreted as pumped AITs, using $p5$ where $x=-1$, $i=-5=7$ and $S=\{3,4\}$.

$$
\begin{aligned}
\{4,5,7,8,9,11\} \quad &\zeta \quad \{6,8,9,10,11,2\} \\
\{4,5,7,11\} \cup \{8,9\} \quad &\zeta \quad \{2,6,8,9\} \cup \{10,11\} \\
5+(\beta \cup \{3,4\}) \quad &\zeta \quad 8-(\beta \cup \{9,10\}) \\
5+(\beta \cup \{3,4\}) \quad &\zeta \quad 8-(\beta \cup 1-\{3,4\})
\end{aligned}
$$

Figure 7: The opening of the third movement of Schoenberg's 3rd String Quartet, where pairs of rows overlap along the [0146]+[05] partition.

Figure 8: The row from Schoenberg's 3rd String Quartet, Op. 30.

a) The row and its partitions.

7	**4**	**3**	**9**	**0**	**5**	**6**	**11**	**10**	**1**	**8**	**2**
[03]		[06]		[05]		[05]		[03]		[06]	
[0146]				[05]		[05]		[0146]			
[012469]						[013468]					

b) Some invariance between row forms.

$\mathbf{P}_7$:	7	4	3	9	0	5	6	11	10	1	8	2
$\mathbf{I}_{10}$:	10	1	2	8	5	0	11	6	7	4	9	3
					–	–	–	–				

$\mathbf{P}_7$:	7	4	3	9	0	5	6	11	10	1	8	2
$\mathbf{R}(\mathbf{I}_{10})$:	3	9	4	7	6	11	0	5	8	2	1	10
	–	–	–	–					–	–	–	–

c) Formula for the two hexachords, using *p*5.

$$\{7,4,3,9,0,5\} \quad \zeta \quad \{6,11,10,1,8,2\}$$
$$3+(\{0,1,4,6\} \bigcup \{2,9\}) \quad \zeta \quad 2-(\{0,1,4,6\} \bigcup 5-\{2,9\})$$

Appendix

Required conditions for Ψ to create valid formulas.

Condition 1:	Ψ must be of a setclass that possesses what we call 'cyclic invariance': the property such that an instance of the setclass is able to transform onto another instance of the same setclass exclusively by transposing individual pitch classes (though not all pcs) by the interval ϕ. This condition reflects the orthogonal nature of z-related sets.
Condition 2:	Ψ should not abstractly include Φ as a subset. (This eliminates possible malformations.)
Condition 3:	Ψ needs to be transposed so that it is symmetrical at index 0 in modϕ (from Callender and Hall 2008).
Condition 4:	Ψ must not be symmetrical at index 0 or m/2 in mod$m$. (Otherwise, the two sides of the formula will be of the same setclass.)

List of sets in mod12 that satisfy the conditions for Ψ, given the cyclic interval $\phi = 6$.

Setclasses with conditions 1 and 2	Possible positioning according to conditions 3 and 4.
[03]	{0, 3}
[024]	{0, 2, 4}
[015]	{0, 1, 5}
[0124]	{11, 0, 1, 3}
[0134]	{11.5, 0.5, 2.5, 3.5}
[0125]	{11.5, 0.5, 1.5, 4.5}
[0135]	{0, 1, 3, 5}
[0235]	{11, 1, 2, 4}
[0237]	{10.5, 0.5, 1.5, 5.5}
[0247]	{11, 1, 3, 6}
[0347]	{10, 1, 2, 5}
[0148]	{11, 0, 3, 7}
[0358]	{0.5, 3.5, 5.5, 8.5}
[01235]	{11, 0, 1, 2, 4}
[01245]	{0, 1, 2, 4, 5}
[02347]	{10, 0, 1, 2, 5}
[02357]	{11, 1, 2, 4, 6}
[01358]	{11, 0, 2, 4, 7}
[01458]	{0, 1, 4, 5, 8}
[012345]	{0, 1, 2, 3, 4, 5}
[023457]	{0, 2, 3, 4, 5, 7}
[013458]	{0, 1, 3, 4, 5, 8}
[024579]	{0, 2, 4, 5, 7, 9}
[014589]	{0, 1, 4, 5, 8, 9}

Example formula:

$$\{0,6\} \cup x + \{0,2,4,5,7,9\} \quad \zeta \quad \{0,6\} \cup -x + \{0,2,4,5,7,9\}$$

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